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Euler Walk on a Cayley Tree

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We describe two possible regimes (dynamic phases) of the Euler walk on a Cayley tree: a condensed phase and a low-density phase. In the condensed phase the area of visited sites grows as a compact domain. In the low-density phase the proportion of visited sites decreases rapidly from one generation of the tree to the next. We describe in detail returns of the walker to the root and growth of the domain of visited sites in the condensed phase. We also investigate the critical behaviour of the model on the line separating the two regimes.

KEY WORDS: Branching processes, critical exponents, martingales, random walks

1. INTRODUCTION

Consider a Cayley tree with arrows attached to every site. Initially the arrows point at one of the adjacent sites randomly and independently of each other, see Fig. 1. An Eulerian walker moves over the Cayley tree according to the following rules. At time instants $l = 0, 1, 2, \ldots$ the walker jumps from its current location $x(l)$ (at one of the sites of the tree) to the adjacent site in the direction of the arrow at $x(l)$. At the time of jump the arrow at $x(l)$ is rotated clockwise, till it points to another adjacent site.

We assume that the generations of the tree are numbered from bottom to top. Zero generation of the tree contains only the root. If $k > l$, then the generation number k is above the generation number l on a picture of the tree, and we say that the kth generation is higher than the generation number l.

Eulerian walkers were introduced by Priezzhev *et al.*[(10)] (see Ref. 9 for further investigations) as a model of "self-organized criticality." In their version of the model the walker rotates arrows on arrival at a site of the graph. The two versions

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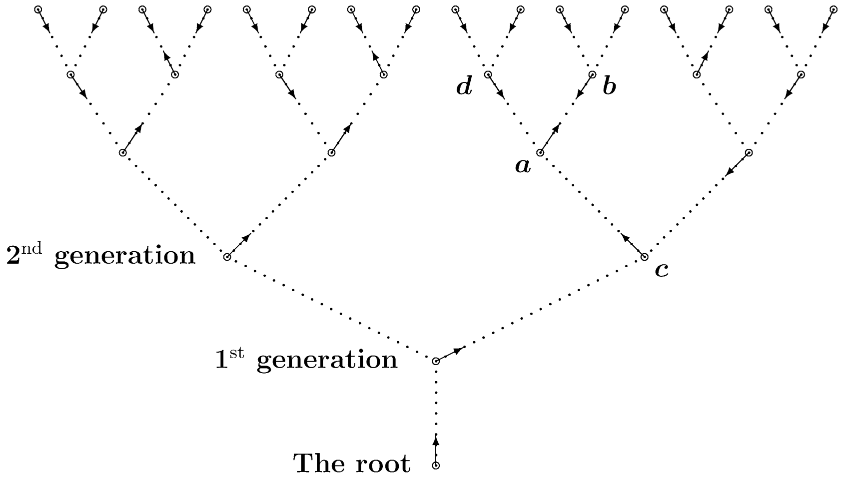

Fig. 1. A Cayley tree with internal arrows arranged at random and independently of each other. Boundary arrows always point inside the tree to prevent the walker from falling off the tree. At site a the walker jumps in the direction of the arrow to site b, and the arrow at a is rotated clockwise to point at site c. After that the walker jumps back to site a and then jumps to site c. At the time of the last jump the arrow at a is again rotated clockwise to point at site d.

are largely equivalent, but in the version used in this paper it is easier to see on the picture of a graph where the walker actually goes over the next few steps.

An attractive feature of the model on a *finite* graph with reflecting boundary (the boundary arrows point inside the graph) is that eventually the walker settles into an Euler circuit, where it passes every edge of the graph twice (once in every direction). There is exactly one "clockwise" Euler circuit for any tree, see Fig. 2. As a result of the walker's activity, initially chaotically oriented arrows arrange into an organized configuration directing the walker around that circuit.

If the Euler circuit on a tree is a critical state is not that clear. It is not difficult to calculate correlation functions for orientations of the arrows at two sites of a Cayley tree, assuming the uniform distribution of the current location of the walker. For instance, let a_1 and a_2 be two arrows at sites in the generations k and $k + m$ of a finite tree containing n generations in total. Then

$$\Pr[a_1 = \searrow, a_2 = \searrow] - \Pr[a_1 = \searrow]\Pr[a_2 = \searrow] \to -2^{-2k-m}, \quad \text{as } n \to \infty.$$

Hence, we have an exponential decay of correlations with the distance between the arrows measured in generations of the tree. The same asymptotic behaviour we obtain for all other correlation functions, although some of those are positive.

The above decay of correlations is in contrast to the behaviour found by Dhar and Majumdar for the self-organized state of a sand pile on a Cayley tree, see Ref. 4. Dhar and Majumdar found that the correlation functions decay as 4^{-m}, where m is

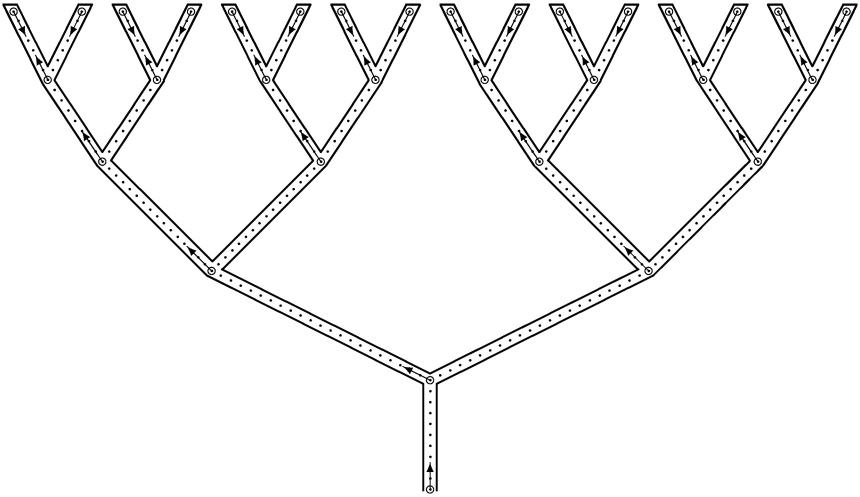

Fig. 2. The Euler circuit on a Cayley tree. The orientation of arrows corresponds to the current position of the walker at the root of the tree.

the distance between the two sites of the tree. They concluded that the correlations are short-ranged, because even after multiplication by the branching factor 2^m one still has an exponential decay to 0. Nevertheless they classified the self-organized state of the sand pile on a Cayley tree as critical, presumably, because of power-law tails in the distributions of avalanche-sizes and related quantities.

In our case the correlations decay as 2^{-m}, and do not vanish after multiplication by the branching factor. Therefore one certainly can not rule out the criticality of the Euler circuit on a tree on the basis of exponential decay of correlation functions. Nevertheless one feels that the self-organized state in this case is closer to the minimally stable state of the 1D sand pile, described in Ref. 2, than to a truly critical state.

Our main goal in this paper is a description of the *formation* of an organized structure on an infinite tree. We will show that, unlike what one sees on finite graphs with reflecting boundary, on an infinite tree a (substantial density of) organized structure is not always formed. Of course, if an organized structure is not formed on an infinite tree, it is highly sensitive to the boundary conditions and appears on a finite tree only as a result of numerous bounces of the walker against the reflecting boundary.

To set the scenery for the study of Euler walk let us describe two possible regimes of evolution: a condensed phase and a low-density phase. By (dynamic) phases in this paper we mean not a particular distribution $P[\boldsymbol{a}]$ of arrows $\boldsymbol{a}$, but a particular *type of evolution* of those distributions $P_t[\boldsymbol{a}]$.

To describe the condensed phase let us arrange all the arrows (except the one at the root of the tree) downwards, along the edges of the tree. In this case the walker starting at the root at time $T_0 = 0$ returns to the root at time instants $T_1 = 2$, $T_2 = 8$, $T_3 = 22, \ldots$. In general, the fth return to the root takes place at the time instant $T_f = 2^{f+2} - 2f - 4$.

There is a growing domain of visited sites—the explored domain—which penetrates the kth generation of the tree at the time instant $t_k \equiv 2^{k+1} - k - 2$, $k = 1, 2, \ldots$. At the time instant $s_k \equiv 2^{k+2} - 3k - 3$ the domain swallows the kth generation completely, and the walker heads toward the root.

If we denote $g_{\max}(t)$ the highest generation visited by the walker by the time t, then the formula for t_k yields

$$\log_2(t) - 1 \le g_{\max}(t) \le \log_2(t), \quad \text{for } t \ge 4.$$

Analogously, if we denote $g_{\mathrm{c}}(t)$ the number of generations completely explored by the time t, then the formula for s_k yields

$$\log_2(t) - 2 \le g_{\mathrm{c}}(t) \le \log_2(t) - 1, \quad \text{for } t \ge 3.$$

Thus, for the downward initial arrangement of the arrows, the growing explored area is a "compact" domain of the tree. The height of the domain (measured in generations) grows with time as $\log_2 t$. Below the highest visited generation the density of visited sites is 1, above that generation the density of visited sites is, of course, 0. This is the *condensed phase* of the Euler walk.

Another regime—*the low-density phase*—is obtained if we begin with the upward (left or right) initial orientation of the arrows. In this case the walker goes straight toward the top of the tree. The density of visited sites in the fth generation at time t is 2^{-f+1} (for $f \le t$), which tends to 0 with f justifying the name the low-density phase. Of course, once the walker reaches the top of the (finite) tree it turns back and gradually stomps the whole graph. Therefore (as it should be) a clear-cut distinction between the two phases exists only on an infinite tree.

For a random initial arrangements of the arrows we obtain a phase which is a perturbation of either the condensed or the low-density phase. As we will see in the following sections, the transition between the two phases takes place when $2\,\Pr[\nwarrow] + \Pr[\nearrow] = 1$.

The rest of the paper is organized as follows. In Sec. 2 we investigate the properties of the condensed phase: the returns of the walker to the root, and the growth of the explored domain when $2\,\Pr[\nwarrow] + \Pr[\nearrow] < 1$. In Sec. 3 we show the absence of a compact domain of visited sites if $2\,\Pr[\nwarrow] + \Pr[\nearrow] > 1$. In Sec. 4 we repeat the program of Sec. 2 at the critical point $2\,\Pr[\nwarrow] + \Pr[\nearrow] = 1$. Traditionally, the last section is devoted to a discussion of the results obtained in the previous sections.

2. THE CONDENSED PHASE

Let the internal arrows be initially arranged independently of one another, and according to the distribution $\Pr[\nwarrow] = p, \Pr[\nearrow] = q, \Pr[\downarrow] = 1 - p - q$. One can map every initial configuration of arrows into a realization of a discrete-time branching process according to the following rules.

Place a particle at the root of the tree. This particle produces exactly one descendant—a particle which is placed at the site of the first generation of the tree. From the first generation on, a particle produces either 0, or 1, or 2 descendants depending on the initial direction of the arrow at the site occupied by the particle. If the arrow points downward, then the particle does not have descendants. If the arrow points up and right (like the arrow at the first-generation site on Fig. 3), then the particle has exactly one descendant placed at the adjacent site in the direction of the arrow. Finally, if the arrow points up and left, then there are exactly two descendants placed at the two adjacent sites above, see Fig. 3.

The relevance of the branching process to our main problem stems from the following fact. If the branching process degenerates, then the walker returns to the root at a finite time-instant T_1 equal twice the number of descendants in the branching process (not counting the original particle at the root). The first-return path encircles the particles in all generations of the branching process, which we call below *the first-return cluster*.

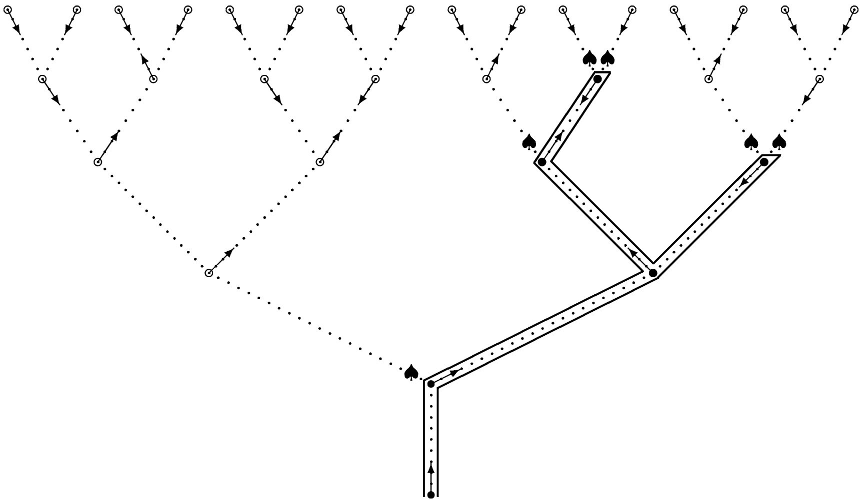

Fig. 3. An initial arrangement of the arrows, the corresponding first return to the root of the Euler walker (*solid lines*), the first-return cluster of the associated branching process (*discs*), and the buds (*spades*). At the next visit to a site with buds an independent first-return cluster will grow from every bud.

The above correspondence between paths of the walker on Cayley tree and realizations of the branching process allows one to employ the elegant technique of generating functions and the main results from the theory of branching processes.(6,5) First of all recall that if a particle produces k descendants with probability p_k, then the branching process degenerates with probability 1 if and only if $\sum_{k=1}^{\infty} kp_k \leq 1$. Hence, the time of the first return is finite with probability 1 if and only if $q + 2p \leq 1$. The critical case $q + 2p = 1$ requires a special consideration, therefore, in this section we consider only the case $q + 2p < 1$.

Lemma 1. Let $q + 2p < 1$, then the walker returns to the root for the first time at an almost surely finite even time-instant T_1, such that

$$m_1 \equiv \boldsymbol{E}T_1 = \frac{2}{1-(q+2p)};$$

$$\operatorname{Var} T_1 = \frac{4(1-q)}{(1-(q+2p))^3} - \frac{4}{1-(q+2p)};$$

$$\Pr[T_1 = 2k] \sim \sqrt{\frac{q\sqrt{(1-p-q)/p} + 2(1-p-q)}{4\pi p}}\, k^{-3/2}\left(q + 2\sqrt{p(1-p-q)}\right)^k,$$

as $k \to \infty$.

Proof: Denote X the number of descendants for a particle outside the root of the tree. The probability generating function of X is given by

$$g(y) \equiv \boldsymbol{E}y^X = 1 - q - p + qy + py^2. \tag{1}$$

Denote Z the total number of descendants in the associated branching process. The probability generating function of Z, $f(x) \equiv \boldsymbol{E}x^Z$, is a solution of the equation, see Refs. 5, 6,

$$f(x) = xg(f(x)).$$

Hence

$$f(x) = \frac{1}{2px}\left[1 - qx - \sqrt{(1-qx)^2 - 4p(1-p-q)x^2}\right].$$

Differentiating $f(x)$ and taking into account $T_1 = 2Z$, we obtain

$$m_1 \equiv \boldsymbol{E}T_1 = \frac{2}{1-(q+2p)}, \quad \operatorname{Var} T_1 = \frac{4(1-q)}{(1-(q+2p))^3} - \frac{4}{1-(q+2p)}.$$

The above generating function $f(x) = \sum_{k=0}^{\infty} p_k x^k$ often appears in the literature on branching processes, see, e.g., the paper(7) by Otter. In particular, it is shown in that paper that the large-k asymptotics for $p_k = P[Z = k]$ is given by

$$p_k \sim \sqrt{\frac{q\sqrt{(1-p-q)/p} + 2(1-p-q)}{4\pi p}}\, k^{-3/2} \left(q + 2\sqrt{p(1-p-q)}\right)^k, \tag{2}$$

which is the announced formula for $P[T_1 = 2k]$ in the statement of this lemma. □

Remark 1. Denote Z_k the number of particles of the associated branching process in the kth generation of the tree. The random variable X is the number of descendants produced by a single particle as in the proof of Lemma 1. Then the distribution of the height of the first-return path, H_1, is given by

$$\Pr[H_1 = k] = \Pr[Z_{k+1} = 0] - \Pr[Z_k = 0].$$

It is shown in the book by Harris$^{(6)}$ that the large-k asymptotics of $\Pr[Z_k = 0]$ is given by

$$\Pr[Z_k = 0] \sim 1 - c_1(\boldsymbol{E}X)^k,$$

if $\boldsymbol{E}X < 1$, where c_1 is an unknown positive constant.

Hence in our case the distribution of H_1 decays exponentially with k,

$$\Pr[H_1 = k] \sim c(q + 2p)^k.$$

Thus, during the first stage of exploration of the Cayley tree ($0 \le t \le T_1$) the walker stomps a first-return path with statistical properties described in Lemma 1. To visualize the motion of the walker after the first return to the root one can imagine that, whenever a site is visited for the first time and X descendants are produced in the associated branching process, the walker attaches $2 - X$ buds to the site, see Fig. 3. During the second stage of exploration (after the first return but before the second return to the root) the walker follows the first-return path, but, whenever a bud is encountered, it wonders off the beaten track and appends to the existing path a new circuit, which (unless hitting the boundary) is statistically equivalent to the first-return path, see Fig. 4.

Lemma 2. Let the walker return to the root for the first time at time T_1. Then the first-return path has exactly $1 + \frac{1}{2}T_1$ attached buds.

Proof: Recall the following standard representation for the number of descendants, Z_k, in generations $k = 2, 3, \dots$ of the associated branching process

$$Z_2 = X_1^{(1)},$$

$$Z_3 = X_1^{(2)} + X_2^{(2)} + \cdots + X_{Z_2}^{(2)},$$

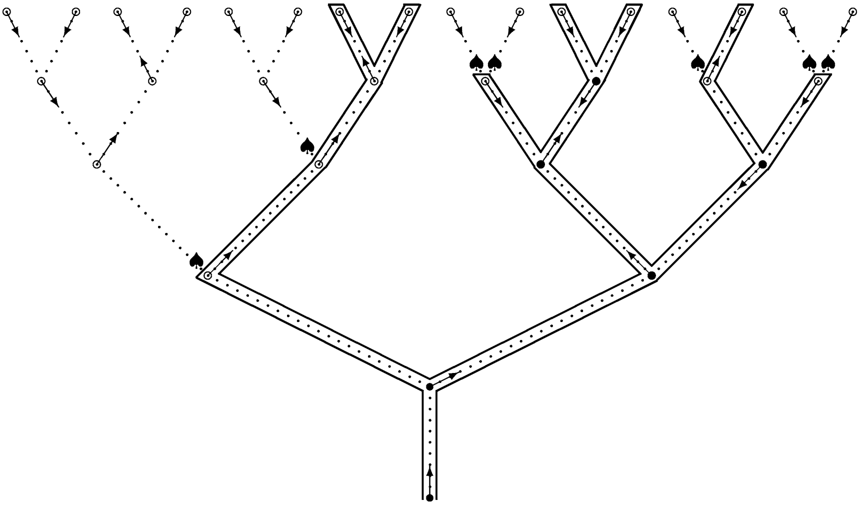

Fig. 4. The initial arrangement of the arrows, the corresponding second return to the root for the Euler walker (*solid lines*), and a new set of buds (*spades*).

$$Z_4 = X_1^{(3)} + X_2^{(3)} + \cdots + X_{Z_3}^{(3)},$$

and so on,

where $X_k^{(l)}$ is the number of descendants produced by the kth particle from the lth generation. All the random variables $X_k^{(l)}$ are independent and have the same distribution as the random variable X. Note also that $Z_1 = 1$, and $Z_{n+1} = 0$ whenever $Z_n = 0$.

Then we have the following formulae for the number of buds b_k, in generations $k = 1, 2, 3, \ldots$

$$b_1 = 2 - X_1^{(1)} = 2 - Z_2,$$

$$b_2 = 2 - X_1^{(2)} + 2 - X_2^{(2)} + \cdots + 2 - X_{Z_2}^{(2)} = 2Z_2 - Z_3,$$

$$b_3 = 2 - X_1^{(3)} + 2 - X_2^{(3)} + \cdots + 2 - X_{Z_3}^{(3)} = 2Z_3 - Z_4,$$

and so on.

Since for $q + 2p < 1$ only a finite number of Z_k have non-zero values, the total number of buds on the first-return path is given by

$$B_1 = \sum_{k=1}^{\infty} b_k = \sum_{k=2}^{\infty}(2Z_{k-1} - Z_k) = 1 + \sum_{k=1}^{\infty} Z_k.$$

The total number of descendants in all generations is $\frac{1}{2}T_1$, hence $B_1 = 1 + \frac{1}{2}T_1$. □

Theorem 1. *Let $q + 2p < 1$, then the Euler walker returns to the root infinitely often at (almost surely finite) time instants $T_1, T_2, T_3, \ldots$. Moreover, the sequence of normalized differences*

$$Y_n = \frac{T_n - T_{n-1} + 2}{\left(1 + \frac{1}{2}\boldsymbol{E}T_1\right)^n}, \quad n = 1, 2, 3, \ldots,$$

is a positive and uniformly integrable martingale, $\boldsymbol{E}[Y_n | Y_{n-1}, \ldots, Y_1] = Y_{n-1}$.

Proof: In order to return to the root for the second time the walker has to repeat the first-return path and to create new first-return circuits at each of the B_1 buds. Hence for the time of the second return to the root we obtain

$$T_2 - T_1 = T_1 - T_0 + \tau_1^{(2)} + \tau_2^{(2)} + \cdots + \tau_{B_1}^{(2)},$$

where $T_0 = 0$, and $\tau_j^{(2)}$ are independent random variables with the same distribution as the first-return time T_1. A verbatim repetition of the argument from the proof of Lemma 2 shows that on each of the new circuits attached to the first-return path the walker creates $1 + \frac{1}{2}\tau_j^{(2)}$ buds, $j = 1, 2, \ldots, B_1$. Hence the total number of buds on the second-return path is given by

$$B_2 = B_1 + \frac{1}{2}\sum_{j=1}^{B_1} \tau_j^{(2)}.$$

We have essentially the same scenario for any return to the root. For the time of the nth return to the root we obtain

$$T_n - T_{n-1} = T_{n-1} - T_{n-2} + \sum_{j=1}^{B_{n-1}} \tau_j^{(n)}. \tag{3}$$

The number of buds on the nth return path is given by

$$B_n = B_{n-1} + \frac{1}{2}\sum_{j=1}^{B_{n-1}} \tau_j^{(n)}.$$

Since $B_1 = 1 + \frac{1}{2}T_1$, we can rewrite the last equation as

$$B_n = \sum_{j=1}^{B_{n-1}} b_j^{(n)}. \tag{4}$$

where $b_j^{(n)} = 1 + \frac{1}{2}\tau_j^{(n)}$ are independent random variables with the same distribution as B_1.

Induction and the obtained relationships for T_n and B_n yield

$$B_n = 1 + \frac{1}{2}(T_n - T_{n-1}), \qquad \text{for any } n \geq 1. \tag{5}$$

Indeed, Lemma 2 says that in the case $n = 1$ this formula is correct. Suppose that the formula is also correct for $n = k$. Then the relationships for B_n and T_n yield

$$\begin{aligned} B_{k+1} = B_k + \frac{1}{2}\sum_{j=1}^{B_k} \tau_j^{(k)} &= 1 + \frac{1}{2}(T_k - T_{k-1}) + \frac{1}{2}(T_{k+1} - T_k - T_k + T_{k-1}) \\ &= 1 + \frac{1}{2}(T_{k+1} - T_k). \end{aligned}$$

Hence $B_n = 1 + \frac{1}{2}(T_n - T_{n-1})$ is also correct for $n = k + 1$, which completes the induction.

Now one can calculate the following conditional expectation

$$\begin{aligned} \boldsymbol{E}\left[T_{n+1} - T_n + 2 | T_n - T_{n-1} + 2\right] &= T_n - T_{n-1} + 2 + \boldsymbol{E}\left[\sum_{j=1}^{B_n} \tau_j^{(n+1)} \middle| T_n - T_{n-1} + 2\right] \\ &= \left(1 + \frac{1}{2}\boldsymbol{E}T_1\right)(T_n - T_{n-1} + 2). \end{aligned}$$

Hence the sequence

$$Y_n = \frac{T_n - T_{n-1} + 2}{\left(1 + \frac{1}{2}\boldsymbol{E}T_1\right)^n}, \qquad n = 1, 2, 3, \ldots,$$

is a positive martingale.

Since $\boldsymbol{E}Y_n = \boldsymbol{E}Y_1 = 2$, we have $\Pr[Y_n < \infty] = 1$, for any n, which implies the almost sure finiteness of the return times T_n, $n = 1, 2, \ldots$.

The relationship $\sup_n \boldsymbol{E}(Y_n^2) < \infty$ is a well-known sufficient condition for the uniform integrability of the sequence $\{Y_n\}_{n=1}^{\infty}$, see, e.g., the book by Shiryaev.$^{(11)}$ In our case Eq. (3) yields

$$\begin{aligned} s_{n+1} &\equiv \boldsymbol{E}(T_{n+1} - T_n + 2)^2 \\ &= \boldsymbol{E}(T_n - T_{n-1} + 2)^2 + 2\boldsymbol{E}\left[(T_n - T_{n-1} + 2)\sum_{j=1}^{B_n} \tau_j^{(n+1)}\right] + \boldsymbol{E}\left(\sum_{j=1}^{B_n} \tau_j^{(n+1)}\right)^2. \end{aligned}$$

On calculating the expected values with the help of the tower property we obtain the following simple recurrent relationship

$$s_{n+1} = s_n \left(1 + \frac{1}{2}\boldsymbol{E}T_1\right)^2 + \mathrm{Var}(T_1)\left(1 + \frac{1}{2}\boldsymbol{E}T_1\right)^n.$$

Solving the recurrent relationship we obtain

$$s_{n+1} = 2\left(2 + \frac{\boldsymbol{E}T_1^2}{\boldsymbol{E}T_1}\right)\left(1 + \frac{1}{2}\boldsymbol{E}T_1\right)^{2n+1} - 2\frac{\mathrm{Var}(T_1)}{\boldsymbol{E}T_1}\left(1 + \frac{1}{2}\boldsymbol{E}T_1\right)^{n}. \qquad (6)$$

Hence

$$\sup_n \frac{s_n}{\left(1 + \frac{1}{2}\boldsymbol{E}T_1\right)^{2n}} < \infty,$$

implying the uniform integrability of the martingale Y_n, $n = 1, 2, \ldots$. □

Corollary 1. Let $q + 2p < 1$, then for almost all initial arrangements of the arrows

$$\lim_{n\to\infty} \frac{T_n - T_{n-1} + 2}{(1 + \frac{1}{2}\boldsymbol{E}T_1)^n} = Y,$$

where Y is a random variable with a proper distribution ($\Pr[Y < \infty] = 1$). The expected value and the variance of the random variable Y are given by

$$\boldsymbol{E}Y = 2, \quad \mathrm{Var}(Y) = \frac{4\mathrm{Var}(T_1)}{\boldsymbol{E}T_1(2 + \boldsymbol{E}T_1)}.$$

Proof: Since the sequence $\{Y_n\}_{n=1}^{\infty}$ is a positive martingale, the Doob martingale convergence theorem, see, e.g., the book by Shiryaev,(11) tells us that $\lim_{n\to\infty} Y_n = Y$, where Y is a random variable with a proper distribution. Since the sequence $\{Y_n\}_{n=1}^{\infty}$ is uniformly integrable $\boldsymbol{E}Y = \lim_{n\to\infty} \boldsymbol{E}Y_n = 2$.

Equation (6) yields

$$\lim_{n\to\infty} \boldsymbol{E}Y_n^2 = 4 + \frac{4\mathrm{Var}(T_1)}{\boldsymbol{E}T_1(2 + \boldsymbol{E}T_1)}.$$

To show that $\boldsymbol{E}Y^2 = \lim_{n\to\infty} \boldsymbol{E}Y_n^2$, we need the uniform integrability of the sequence $\{Y_n^2\}_{n=1}^{\infty}$. To that end one can use the sufficient condition $\sup_n \boldsymbol{E}Y_n^3 = \sup_n \boldsymbol{E}(Y_n^2)^{3/2} < \infty$. One can check by a direct calculation similar to that used in the proof of Theorem 1 that the sufficient condition is indeed satisfied. Hence

$$\mathrm{Var}(Y) = \boldsymbol{E}Y^2 - (\boldsymbol{E}Y)^2 = \frac{4\mathrm{Var}(T_1)}{\boldsymbol{E}T_1(2 + \boldsymbol{E}T_1)}.$$

□

Corollary 2. Let $q + 2p < 1$, then

$$\boldsymbol{E}T_n = 2\,\frac{2 + \boldsymbol{E}T_1}{\boldsymbol{E}T_1}\left[\left(1 + \frac{1}{2}\boldsymbol{E}T_1\right)^n - 1\right] - 2n,$$

$$\mathrm{Var}(T_n) \sim \frac{\mathrm{Var}(T_1)}{\left(\frac{1}{2}\boldsymbol{E}T_1\right)^3}\left(1+\frac{1}{2}\boldsymbol{E}T_1\right)^{2n+1}, \tag{7}$$

$$\lim_{n\to\infty}\frac{T_n}{\left(1+\frac{1}{2}\boldsymbol{E}T_1\right)^{n+1}} = \frac{2Y}{\boldsymbol{E}T_1}, \quad \text{almost surely,}$$

where the random variable Y is identical to the one from Corollary 1.

Proof: Recall that the martingale $\{Y_k\}_{k=1}^{\infty}$ is defined by

$$Y_k = \frac{T_k - T_{k-1} + 2}{\left(1+\frac{1}{2}\boldsymbol{E}T_1\right)^k}.$$

Taking the denominator to the l.h.s. and summing over k from 1 to n one obtains

$$T_n = \sum_{k=1}^{n}\left(1+\frac{1}{2}\boldsymbol{E}T_1\right)^k Y_k - 2n, \tag{8}$$

where we have used $T_0 = 0$. Since $\boldsymbol{E}Y_l = 2$, a summation of the geometric series yields

$$\boldsymbol{E}T_n = 2\,\frac{2+\boldsymbol{E}T_1}{\boldsymbol{E}T_1}\left[\left(1+\frac{1}{2}\boldsymbol{E}T_1\right)^n - 1\right] - 2n.$$

Equation (8), the martingale property $\boldsymbol{E}(Y_k|Y_f) = Y_f$, for $f < k$, and straightforward calculations yield the main asymptotics of the variance $\mathrm{Var}(T_n)$, Eq. (7).

Since $\lim_{k\to\infty} Y_k = Y$ (almost surely), an application of a standard technique from analysis to Eq. (8) yields

$$\lim_{n\to\infty}\frac{T_n}{\left(1+\frac{1}{2}\boldsymbol{E}T_1\right)^{n+1}} = \lim_{n\to\infty}\sum_{k=1}^{n}\left(1+\frac{1}{2}\boldsymbol{E}T_1\right)^{k-n-1} Y_k = \frac{2Y}{\boldsymbol{E}T_1}.$$

□

The last Corollary describes in detail the large-n behaviour of the nth return time T_n in the subcritical regime, where $\boldsymbol{E}T_1 < \infty$. The following crude bound will be helpful at the critical point.

Corollary 3.

$$T_n - T_{n-1} \le T_n \le 2(T_n - T_{n-1}). \tag{9}$$

Proof: For the number of buds on the nth return (to the root) path we have $B_n \geq 2B_{n-1}$, hence $B_{n-l} \leq 2^{-l}B_n$. Summing Eq. (5) we obtain

$$T_n = 2\left(\sum_{l=1}^{n} B_l - n\right).$$

Therefore $T_n \leq 4B_n - 2n$, and using Eq. (5) again we obtain

$$T_n - T_{n-1} \leq T_n \leq 2(T_n - T_{n-1}).$$

□

Theorem 1 and its corollaries give a fairly comprehensive description of the frequency of return to the root. Our next aim is a description of the height of the domain of visited sites. Remark 1 describes the distribution of the highest visited generation at time T_1. Investigation of the height of the domain at later times is a much more delicate problem. We will find the asymptotic behaviour of the density of visited sites, $v_k(t)$, in the kth generation of the tree, defined as the ratio of the number of sites visited by time t to the total number of sites, 2^{k-1}, in the kth generation.

In order to describe the growth of the domain of visited sites on the Cayley tree, let us consider an arbitrary branch $w_n = (e_1, e_2, \ldots, e_n)$ of the tree, where e_l, $l = 1, 2, \ldots, n$ are the segments (edges) of the branch, see Fig. 5. With any edge

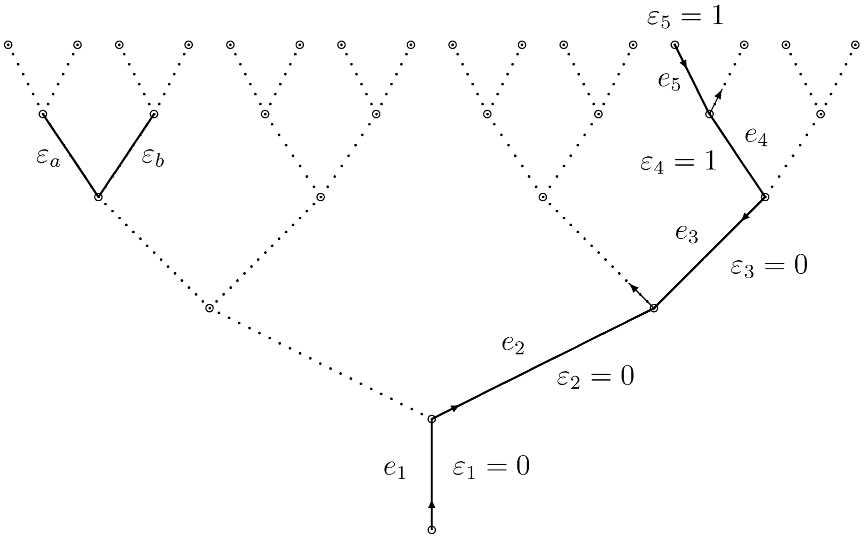

Fig. 5. A branch w_5 (path) of the Cayley tree, its edges $(e_1, e_2, \ldots, e_5)$, and the associated random "energies" $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_5)$. The energies of edges growing from the same site of the tree, like ε_a and ε_b, are not independent.

e_l one can associate an "energy" ε_l as follows. The energy of a left edge e_l (like e_a on Fig. 5) is equal to 0, if the arrow at the bottom of the edge e_l points along the edge, and $\varepsilon_k = 1$ otherwise. The energy of a right edge e_l (like e_b on Fig. 5) is equal to 1, if the arrow at the bottom of the edge e_l points down, and $\varepsilon_l = 0$ otherwise. In other words, the energy of an edge e_l is equal to 1, if the arrow at the bottom of the edge causes the walker to deviate from the Euler circuit, and the energy is equal to 0 if the walker passes the edge "effortlessly." The energy of a branch w_n is the sum of the energies of its edges.

The domain of visited sites swallows up the edges of a path w_n as follows. During the time interval $[0, T_1]$ (before the first return to the root) the domain swallows all the edges of the path w_n till the first obstacle—the first edge e_l with $\varepsilon_l = 1$. During the time interval $[T_1, T_2]$ (after the first return but before the second return to the root) the domain of visited sites swallows up the edge e_l and all zero-energy edges which follow e_l until the second obstacle—the second edge e_m with non-zero energy, and so on. During the time interval $[T_j, T_{j+1}]$ (after the jth return but before the $j+1$th return to the root) the domain of visited sites swallows up all the edges between the jth and $j+1$th edges with non-zero energy. Thus, the number of visited sites in the kth generation at time T_m is equal to the number of paths w_k with less than m obstacles, or, equivalently, with the path energies $E(w_k) = \sum_{l:e_l \in w_k} \varepsilon_l$ less than m.

Let us consider the following sum (partition function)

$$\Theta_k = \sum_{w_k} \exp[-\beta E(w_k)],$$

where the summation runs over all branches w_k of a tree with k generations. We have

$$\Theta_k = \sum_{n=0}^{k} \#\{w_k : E(w_k) = n\} \exp[-\beta n].$$

Hence, the large k limit of $k^{-1} \ln \Theta_k$ is the Legendre-transform of

$$\nu(y) \equiv \lim_{k \to \infty} k^{-1} \ln \#\{w_k : E(w_k) = [ky]\},$$

where $[ky]$ is the integer part of ky.

On the other hand, the sum Θ_n is almost identical to the partition function of a directed polymer on a Cayley tree, see Ref. 3. The difference between Θ_n and the partition function in Ref. 3 is that not all the energies ε_l are independent. Indeed if two edges e_a and e_b grow from the same site of the tree, see Fig. 5, then

$$\Pr[\varepsilon_a = 1, \varepsilon_b = 1] = 1 - p - q, \quad \Pr[\varepsilon_a = 0, \varepsilon_b = 1] = 0,$$

$$\Pr[\varepsilon_a = 1, \varepsilon_b = 0] = q, \quad \text{and} \quad \Pr[\varepsilon_a = 0, \varepsilon_b = 0] = p.$$

Nevertheless, the large-k asymptotics of $k^{-1}\ln\Theta_k$ can be found by virtually verbatim repetition of the derivation from Ref. 3. In particular, if we denote $\mathcal{A}_k$ the σ-algebra generated by the random energies of the first k generations of the tree, and define

$$M_k = \frac{\Theta_k}{[(2-2p-q)e^{-\beta}+2p+q]^{k-1}},$$

then the stochastic sequence $\{M_k, \mathcal{A}_k\}_{k=1}^{\infty}$ is a positive martingale, and $\boldsymbol{E}M_k = 1$.

Using the martingale technique from Ref. 3 we obtain.

Proposition 1. *If* $0 \le 2p+q < 1$, *then*

$$f(\beta) \equiv \lim_{k\to\infty} k^{-1}\ln\Theta_k = \begin{cases} \ln[(2-2p-q)e^{-\beta}+2p+q], & \text{if } \beta \le \beta_c; \\ \dfrac{\beta}{\beta_c}\ln[(2-2p-q)e^{-\beta_c}+2p+q], & \text{if } \beta \ge \beta_c; \end{cases} \tag{10}$$

where β_c *is the positive solution of*

$$\ln[(2-2p-q)e^{-\beta}+2p+q] = \frac{\beta(2p+q)e^{\beta}}{2-2p-q+(2p+q)e^{\beta}}.$$

While if $1 \le 2p+q \le 2$, *then*

$$f(\beta) \equiv \lim_{k\to\infty} k^{-1}\ln\Theta_k = \ln[(2-2p-q)e^{-\beta}+2p+q]. \tag{11}$$

Lemma 3. The logarithmic asymptotics of the number of path w_k with the energy $[ky]$, $y \in (0,1)$ is given by

$$\begin{aligned} \nu(y) &\equiv \lim_{k\to\infty} k^{-1}\ln\#\{w_k : E(w_k) = [ky]\} \\ &= \left[y\ln\frac{2-(2p+q)}{y} + (1-y)\ln\frac{2p+q}{1-y}\right]^{+}, \end{aligned} \tag{12}$$

where $[x]^{+} = \max(x,0)$ is the positive part of x.

Proof: The free energy $f(\beta)$, given by Eqs. (10) and (11), is the Legendre transform of the logarithmic asymptotics $\nu(y)$. Namely

$$f(\beta) = \max_{y\in[0,1]}[-\beta y + \nu(y)].$$

Therefore

$$\nu(y) = \min_{\beta\ge 0}[\beta y + f(\beta)].$$

Solving the minimization problem we obtain Eq. (12). □

An inspection of the function $\nu(y)$ shows that there are around $[2-(2p+q)]^k$ branches w_k containing k obstacles for the walker to overcome. At the same time there are a few branches with only around $[y^*k]$ obstacles, where $y^* \in (0,1)$ is a solution of the equation

$$y \ln \frac{2-(2p+q)}{y} + (1-y)\ln \frac{2p+q}{1-y} = 0.$$

Hence, there exists a growing with time gap, of the width $m(1/y^* - 1)$ generations at time T_m, between the highest visited generation and the highest completely explored generation of the Cayley tree. Therefore, neither generation is likely to be a sensible measure of the height of the domain of visited sites.

It is a common practice in situations like that to concentrate ones attention on typical branches of the tree. Therefore, we define the height of the domain of visited sites as a number (function) $H(t) \sim h \ln t$, such that the density of visited sites in generation $x \ln t$ at time t, $v_{x \ln t}(t)$, tends to zero with t if $x > h$, and $v_{x \ln t}(t) \to 1$, if $x < h$. We will see shortly that this definition is a sensible one for the problem under consideration. Of course, the choice of the asymptotic form $H(t) \sim h \ln t$ is specific to Cayley trees, and was actually made after the density of visited sites was calculated.

The logarithmic asymptotics $\nu(y)$ attains its maximum, $\ln 2$, at $y = p + \frac{1}{2}q$. Hence, the typical branches w_k have the energy $E(w_k) \sim k(p + \frac{1}{2}q)$. Thus, the domain of visited sites swallows up a typical branch w_k of the tree after $k(p+\frac{1}{2}q)$ returns to the root.

Theorem 2. *Let $q + 2p < 1$, then the height of the domain of visited sites, $H(t)$, grows as logarithm of time,*

$$H(t) \sim \frac{\ln t}{\left(p + \frac{1}{2}q\right)\ln\left(1 + \frac{1}{2}\boldsymbol{E}T_1\right)}.$$

Proof: As follows from Corollary 2, the number of returns to the root by time t for the walker is given by

$$m \sim \frac{\ln t}{\ln\left(1 + \frac{1}{2}\boldsymbol{E}T_1\right)},$$

as $t \to \infty$. The asymptotic number of obstacles in a typical branch w_k of the Cayley tree is given by $k(p+\frac{1}{2}q)$, as $k \to \infty$. Hence, the typical penetration after m returns to the root is approximately $m/(p+\frac{1}{2}q)$ generations, while the typical penetration by time t is

$$H(t) \sim \frac{\ln t}{(p + \frac{1}{2}q)\ln\left(1 + \frac{1}{2}\boldsymbol{E}T_1\right)} \quad \text{generations.}$$

□

Unfortunately it is difficult to go beyond the logarithmic asymptotics $\nu(y)$ of the number of paths w_k with the energy $E(w_k) = [ky]$. Nevertheless, one can guess that the number of paths with the energy $E(w_k) \sim k(p + \frac{1}{2}q) + \sqrt{k}u$ is controlled entirely by the quadratic term in the Taylor expansion for $\nu(y)$ at $y = p + \frac{1}{2}q$. If this is indeed the case then, in the spirit of the local limit theorem, we obtain

$$\#\left\{w_k : E(w_k) = k\left(p+\frac{1}{2}q\right) + \sqrt{k}u\right\} \sim \frac{c}{\sqrt{k}} \exp\left[k\nu\left(p + \frac{1}{2}q\right) + \frac{1}{2}\nu''\left(p + \frac{1}{2}q\right)u^2\right]$$

$$= \frac{2^k c}{\sqrt{k}} \exp\left[-\frac{u^2}{2(p + \frac{1}{2}q)(1 - p - \frac{1}{2}q)}\right]. \tag{13}$$

The density of visited sites in generation n at time T_m is given by

$$v_n(T_m) = \frac{1}{2^{n-1}} \sum_{f<m} \#\{w_n : E(w_n) = f\}.$$

Approximating the sum by an integral (very much like in the normal approximation to the binomial distribution) and taking into account Eq. (13) one obtains

$$v_n(T_m) \sim \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{[m-n(p+q/2)]/\sqrt{n}} dx\, \exp\left(-\frac{x^2}{2\sigma^2}\right), \tag{14}$$

where $\sigma^2 = (p + \frac{1}{2}q)(1 - p - \frac{1}{2}q)$.

We summarize the above discussion by a hypothesis which might well be true.

Hypothesis 1. The width of the boundary of the domain of visited sites of size n generations grows with n as $\sqrt{n}$. The drop of the density of visited sites on the boundary from 1 to 0 is described by the error function, see Eq. (14).

Note that at the critical point $2p + q = 1$ the variance σ^2 in Eq. (14) reaches its maximal value, $\frac{1}{4}$, but remains finite. Therefore the density profile of the domain of visited sites does not disintegrate as we approach the critical point. Instead, as $2p + q$ approaches 1, the walker tends to spend more and more time in long (low-density) excursions away from the compact domain of visited sites. Those long excursions do not create new compact visited domains, somewhat like water poured into sand does not create puddles.

3. THE LOW-DENSITY PHASE

Let now $q + 2p > 1$. In this case the associated branching process degenerates with probability x^* which is a solution of the equation $x = g(x)$ less than 1, see Refs. 5, 6, where the function $g(x)$ is given by Eq. (1). That is, $x^* = (1 - q - p)/p$.

A routine application of the Borel-Cantelli lemma shows that in this case, with probability 1, the Euler walker visits the root (and any given generation of the tree) only a finite number of times.

Let k be large enough to guarantee that only one copy of the associated branching process—the copy which does not degenerate—has survived until the kth generation. Then the number of visited sites in the kth generation, V_k, (after the last visit of the kth generation) does not exceeds the number of particles in a single copy of the associated branching process. Namely, $V_k \leq W(q+2p)^k$, where W is a random variable with a proper distribution ($P[W < \infty] = 1$). Since $q + 2p < 2$ unless $p = 1$, we have $V_k/2^k \to 0$ as $k \to \infty$. That is, the model is in the low-density phase when $q + 2p > 1$.

The bound $V_k \leq W(q+2p)^k$ is a gross overestimation of the number of visited sites. Most likely V_k does not grow faster than something like a constant times $\ln k$.

4. THE CRITICAL POINT

In this section we consider the critical case $q + 2p = 1$. Like in the subcritical case $q + 2p < 1$, the associated branching process degenerates with probability 1 if $q + 2p = 1$. However the branching process becomes critical, and its properties differ substantially from those in the subcritical regime. As we shall see shortly, the first moments of all relevant random variables are infinite if $q + 2p = 1$. As a consequence, extraction of properties of the random variables from their generating functions is no longer straightforward.

Lemma 4. Let $q + 2p = 1$, then the walker returns to the root for the first time at a finite (almost surely) time-instant T_1, such that

$$\Pr[T_1 = 2k] \sim \frac{1}{2\sqrt{\pi p}}\, k^{-3/2}, \quad \text{as } k \to \infty. \tag{15}$$

Proof: Analogously to the subcritical case, the probability generating function of the total number of descendants, Z, is given by

$$f(x) = 1 + \frac{1}{2px}\left[1 - x - \sqrt{(1-x)[1-(1-4p)x]}\right]. \tag{16}$$

Using Eq. (2) we obtain

$$\Pr[T_1 = 2k] = \Pr[Z = k] \sim \frac{1}{2\sqrt{\pi p}}\, k^{-3/2}, \quad \text{as } k \to \infty.$$

□

Remark 2. The large-k asymptotics of $\Pr[T_1 = 2k]$ makes it clear that $\boldsymbol{E}T_1 = \infty$. It is still desirable to have a deterministic measure indicating likely values of the first-return time T_1. For that purpose one can use the quantiles $Q_1(x)$—solutions of the equation $\Pr[T_1 \leq Q_1] = x$. The asymptotic formula (15) yields the following equation for approximate values of $Q_1(x)$

$$\frac{1}{2\sqrt{\pi p}} \sum_{k>Q_1/2}^{\infty} k^{-3/2} = 1 - x.$$

Replacing the sum by an integral and solving the obtained equation for $Q_1(x)$ one obtains $Q_1(x) \approx \frac{2}{\pi p(1-x)^2}$. For values of x close to 1, the precision of the found approximation for $Q_1(x)$ is quite reasonable. For instance, in the case $p = 0.1$ it gives $Q_1(\frac{3}{4}) \approx 102$, while the exact value is $Q_1(\frac{3}{4}) = 98$.

Remark 3. Like in the subcritical case, see Remark 1, asymptotic properties of the distribution of the height of the first-return path, H_1, follow from standard results of the theory of branching processes. It is shown in the book by Harris$^{(6)}$ that the large-k asymptotics of $\Pr[Z_k = 0]$ in the case $\boldsymbol{E}X = 1$ is given by

$$1 - \Pr[Z_k = 0] \sim \frac{1}{pk}.$$

Hence, the distribution of H_1 displays a power-law decay,

$$\Pr[H_1 = k] = \Pr[Z_{k+1} = 0] - \Pr[Z_k = 0] \sim \frac{1}{pk^2}.$$

In order to investigate the distribution of the return to the root instants $T_2, T_3, \ldots$ let us first find the probability generating functions $G_2(x), G_3(x), \ldots$ for the number of buds $B_2, B_3, \ldots$ on the corresponding paths. Using Eq. (4) and the tower property one obtains

$$G_n(x) = \boldsymbol{E}x^{B_n} = \boldsymbol{E}(xf(x))^{B_{n-1}} = G_{n-1}(\varphi(x)),$$

where $\varphi(x) \equiv xf(x)$ is the generating function of B_1, and $f(x)$ is given by Eq. (16). It is clear now that $G_n(x)$ is the nth iteration of $\varphi(x)$, that is,

$$G_n(x) = \underbrace{\varphi(\varphi(\ldots\varphi(x)\ldots))}_{n \text{ times}}.$$

Hence $G_n(x) = \varphi(G_{n-1}(x))$ as well.

Theorem 3. *Let $q + 2p = 1$, then the walker returns to the root infinitely often at (almost surely) finite time instants $T_1, T_2, T_3, \ldots$. Moreover*

$$\Pr[T_n - T_{n-1} = 2k] \sim \frac{1}{2^n \Gamma(1 - 2^{-n}) p^{1-2^{-n}} k^{1+2^{-n}}} \quad \textit{as } k \to \infty. \tag{17}$$

Proof: The probability $\Pr[T_n - T_{n-1} = 2k]$ is given by the integral

$$\Pr[T_n - T_{n-1} = 2k] = \frac{1}{2\pi i} \int_C \frac{G_n(z)}{z^{k+2}}\, dz,$$

where C is a sufficiently small closed contour encircling 0, and $G_n(x)$ is the probability generating function of B_n. To find the large-k asymptotics of this integral we adapt the contour integration from Ref. 7. For that purpose we have to know analytical properties of the generating functions $G_n(x)$.

By definition

$$G_n(z) = \sum_{l=0}^{\infty} \Pr[B_n = l] z^l,$$

hence the function $G_n(z)$ is analytic inside the unit circle $\{z : |z| < 1\}$. Since $G_{n-1}(1) = 1$, and $G_n(z) = \varphi(G_{n-1}(z))$, the point $z = 1$ is a branch point of $G_n(z)$. Since $|G_n(e^{ix})| < 1$ for any real $x \in (0, 2\pi)$, the point $z = 1$ is the only singularity of the function $G_n(z)$ on the boundary of the unit circle $\{z : |z| < 1\}$.

From the explicit formula for the function $\varphi(z)$ it is clear that the generating function $G_n(z)$ has only a finite number of points of non-analyticity. Hence, there exists a disc $A_n = \{z : |z| \leq \alpha_n\}$, with $\alpha_n > 1$, such that $z = 1$ is the only singularity of the functions $G_k(z)$, $k = 1, 2, \ldots, n$ in A_n.

Denote D_n the boundary of the disc A_n with a radial cut running outwards from $x = 1$. The generating function $G_n(z)$ can be written as follows

$$G_n(z) = 1 - a_n(1 - z)^{2^{-n}} + (1 - z)^{2^{-n+1}} f_n(z), \tag{18}$$

where $f_n(z)$ is analytic and bounded inside D_n: $|f_n(z)| \leq b(p) < \infty$. Indeed, we already know that the function $G_n(z)$ is analytic inside D_n. Since

$$f_n(z) = \frac{G_n(z) - 1 + a_n(1 - z)^{2^{-n}}}{(1 - z)^{2^{-n+1}}},$$

it must be analytic inside D_n as well.

To show that $f_n(z)$ is bounded inside D_n we can use induction. The function $f_1(z)$ is obviously bounded in any circle with finite radius. Assume now that $f_n(z)$ is bounded in any circle with finite radius for $n = k$, then for $n = k + 1$ we obtain

$$G_{k+1}(z) = \varphi(G_k(z)) = 1 - \sqrt{\frac{a_k}{p}}(1 - z)^{2^{-k-1}} + (1 - z)^{2^{-k}} f_{k+1}(z),$$

where

$$f_{k+1}(z) = \left(\frac{1}{2p} - 1\right)\left[a_k - (1-z)^{2^{-k}} f_k(z)\right]$$

$$-(1-z)^{-2^{-k-1}}\left[\frac{1}{2p}\sqrt{\left[a_k - (1-z)^{2^{-k}} f_k(z)\right][1-(1-4p)G_k(z)]} - \sqrt{\frac{a_k}{p}}\right].$$

Hence $f_{k+1}(z)$ is bounded in any circle with finite radius as well, completing the induction.

From the above equations we obtain the recurrent relationship $a_{k+1} = \sqrt{a_k p^{-1}}$, with the initial condition $a_1 = \sqrt{p^{-1}}$. The solution of this recurrent relationship is given by $a_k = p^{-1+2^{-k}}$.

On substitution of Eq. (18) in the integral representation for the probability $\Pr[T_n - T_{n-1} = 2k]$ we obtain

$$\Pr[T_n - T_{n-1} = 2k] = -\frac{a_n}{2\pi i}\int_C \frac{(1-z)^{2^{-n}}}{z^{k+2}}\,dz + \frac{1}{2\pi i}\int_C \frac{(1-z)^{2^{-n+1}} f_n(z)}{z^{k+2}}\,dz =$$

$$= (-1)^k a_n \binom{2^{-n}}{k+1} + \frac{1}{2\pi i}\int_{D_n} \frac{(1-z)^{2^{-n+1}} f_n(z)}{z^{k+2}}\,dz.$$

Since the function $f_n(z)$ is bounded inside D_n, the remaining integral is of the same order as

$$\int_1^{\alpha_n} \frac{(1-x)^{2^{-n+1}}}{x^{k+2}}\,dx = O\binom{2^{-n+1}}{k+1}.$$

Therefore

$$\Pr[T_n - T_{n-1} = 2k] \sim \frac{1}{2^n \Gamma(1-2^{-n}) p^{1-2^{-n}} k^{1+2^{-n}}} \quad \text{as } k \to \infty.$$

Finally, note that $\Pr[T_n - T_{n-1} < \infty] = 1$, and according to Eq. (9) we have $T_n \leq 2(T_n - T_{n-1})$. Hence, all return to the root instants T_n are almost surely finite. □

Theorem 4. *Let $q + 2p = 1$, then the median of the height of the domain of visited sites grows with time as the iterated logarithm $2\log_2\log_2 t$.*

Proof: If $q + 2p = 1$, then the number of obstacles in a typical branch w_k of a Cayley tree is $\sim k/2$. Therefore it takes $\sim k/2$ returns to the root for the domain of visited sites to reach the kth generation of the tree.

The median $m(n)$ of the duration of nth return loop satisfies

$$\sum_{k>m(n)/2}^{\infty} \Pr[T_n - T_{n-1} = 2k] \sim \sum_{k>m(n)/2}^{\infty} \frac{1}{2^n \Gamma(1-2^{-n}) p^{1-2^{-n}} k^{1+2^{-n}}} = \frac{1}{2}.$$

Replacing the sum by an integral and solving the equation for $m(n)$, we obtain

$$m(n) \sim 2^{2^n} c,$$

as $n \to \infty$. That is, with probability $\frac{1}{2}$, it takes over $2^{2^n} c$ time units for the walker to complete the nth return path.

According to Corollary 3

$$T_n - T_{n-1} \leq T_n \leq 2(T_n - T_{n-1}).$$

Hence the median of T_n is between $2^{2^n} c$ and $2^{2^n+1} c$ once n is sufficiently large.

The kth generation of the tree is reached with probability $\frac{1}{2}$ at a time $t \sim c\, 2^{2^{k/2}}$. Solving the equation $t = c\, 2^{2^{k/2}}$ for k, we obtain

$$k \sim 2 \log_2 \log_2 t,$$

as $t \to \infty$. □

5. DISCUSSION AND CONCLUDING REMARKS

The analysis of the previous sections can be generalised to the case of a Cayley tree with the branching ratio $b > 2$ at the expense of extra technical efforts. Let the arrow directions at every site be numbered counterclockwise $0, 1, 2, \ldots, b$ starting from the direction towards the root. Let also $\Pr[X = k] = p_k$, $k = 0, 1, \ldots, b$ be the initial distribution of arrow directions at every site of the tree. Then we can associate a realization of a branching process to every initial configuration of arrows as follows. From the first generation on, a particle of the associated branching process at a particular site of the tree has $k \in \{0, 1, \ldots, b\}$ descendants if the arrow at that site points in the direction number k. The new particles are placed immediately above the parent at the adjacent sites in the directions $1, 2, \ldots, k$.

The associated branching process is critical if $\sum_{k=1}^{b} k p_k = 1$. Already for $b = 3$ the explicit formula for the generating function of the total number of particles in the branching process, $f(x) = \boldsymbol{E} x^Z$, becomes very cumbersome. For $b > 4$ we lose the luxury of explicit formulae completely. Nevertheless, the results of Lemma 1 are not difficult to derive for the case of general b. For the condensed phase the formulae for differentiation of implicit functions yield

$$m_1 \equiv \boldsymbol{E} T_1 = \frac{2}{1 - \sum_{k=1}^{b} k p_k};$$

$$\operatorname{Var} T_1 = \frac{4\sum_{k=0}^{b}(k-1)^2 p_k}{\left(1-\sum_{k=1}^{b} k p_k\right)^3} - \frac{4}{1-\sum_{k=1}^{b} k p_k}.$$

The tail of the first-return probability, $P[T_1 = 2k]$, can be described in terms of a positive solution, x^*, of the equation

$$\sum_{k=2}^{b} p_k (k-1) x^k = p_0.$$

Namely, see Ref. 7,

$$\Pr[T_1 = 2k] \sim \sqrt{\frac{f(x^*)}{2\pi f''(x^*)}}\, k^{-3/2} \left(\frac{f(x^*)}{x^*}\right)^k,$$

as $k \to \infty$.

We see that the properties of the first-return time for $b > 2$ are qualitatively similar to the analogue results in the case of the branching ratio 2. In a similar way all the conclusions of the previous sections can be generalized to the case $b > 2$, and the generalization does not produce a novel behaviour.

Of course the Euler walk on a Cayley tree is only a toy version of Euler walks on 2D or 3D lattices. Nevertheless we believe/hope that some of the main features of the Euler walk described in this paper are also present in finite-dimensional cases. In particular, we believe that finite-dimensional walks also have the condensed and the low-density phases, and a transition between them.

Martingales might prove to be also useful for investigation of the finite-dimensional walks, but in what way and to what extent is yet to be discovered. Some general properties of the growth of the domain of visited sites on 2D lattices might be similar to those found in the present paper. In particular, the drop of density from 1 to 0 in 2D case might still be described by the error function, cf. Eq. (14). The relationship between the size of domain and fluctuations of its boundary might still be the same square-root law as in Hypothesis 1. It is possible to state a few more similar hypothesis, however, the last one already sounds very bold, and it might be dangerous to continue any further. In any case, analytical investigation of the growth of domain of visited sites for finite-dimensional lattices looks like a very tough problem indeed.

Monte Carlo simulations for square lattices with *equally likely* initial directions of arrows at every site were conducted in Refs. 9 and 10. The simulations show that for 2-D square lattice the radius of the domain of visited sites, $R(t)$, grows with time as $R(t) \sim c\, t^{1/3}$. It was also conjectured that on a 3-D square lattice and in higher dimensions we have a diffusive behaviour, $R(t) \sim c\, t^{1/2}$, because the walker does not return to the cluster of visited sites frequently enough. The last conjecture seems to imply that as the lattice dimension tends to infinity the behaviour of the walker does not become more and more similar to that of a walker on a Cayley tree. However, on the basis of results obtained in this paper one can put forward the following alternative interpretation of the "diffusive" behaiviour of the walker on a 3-D lattice. It might be the case that the Monte Carlo simulations for 3-D lattices were simply conducted in the low-density phase where a compact domain of visited sites is not formed. Changing the initial distribution of arrows one can get into the condensed phase, where the radius of the domain of visited sites grows, presumably, as $R(t) \sim c_d\, t^{1/(d+1)}$, converging to the (Cayley tree) logarithmic behaviour as the lattice dimension $d \to \infty$.

Something similar actually happens on Cayley trees as well. If we take a Cayley tree with the branching ratio $b = 2$, then the equally likely initial distribution of arrows $p_0 = p_1 = p_2 = \frac{1}{3}$ puts the walker at the critical point $p_1 + 2p_2 = 1$. While if we increase the branching ratio to 3, then the equally likely distribution $p_0 = p_1 = p_2 = p_3 = \frac{1}{4}$ corresponds to the low-density phase $p_1 + 2p_2 + 3p_4 > 1$.

It was already known that branching processes are relevant to and, in fact, provide a mean-field description for some model of self-organized criticality, see, e.g. Refs. 1, 12. Although branching process are also relevant to Euler walks, the latter apparently belong to a somewhat different class of models, since instead of fixed values for the standard set of critical exponents, we have a whole spectrum of those. Indeed, instead of the mean-field exponent $\tau = 3/2$, describing the distribution of the size of avalanches, we have the sequence $\tau_n = 1 + 2^{-n}$, $n = 1, 2, \ldots$, beginning with $3/2$.

Due to the infinite memory of the Euler walk it is difficult to calculate the moments of the walker's location, $\boldsymbol{E}x^k(t)$. It is a pity, since the second moment of the walker's location for the simple random walk on, say, 2-D lattices, $\boldsymbol{E}x^2(t) = ct$, is one of the main characteristics of that random process. To partially fill this gap we will extract some information on the behaviour of the second moment from the results obtained in the previous sections. This information might provide clues for explanation of a bizarre behaviour of $\boldsymbol{E}x^2(t)$ for certain versions of Euler walk on 2-D lattices.(8) It is instructive to compare at the same time the behaviour of the Euler walk on a Cayley tree and the simple random walk on a 2-D square lattice.

Both the Euler walk on a Cayley tree in the condensed phase and the 2-D simple random walk are recurrent. Here, however, similarities end. While the expected return-time (and even the variance) for the Euler walk is finite, the expected return-time for 2-D random walk is infinite. As a consequence we have monotonically increasing variance of the walker's location for the 2-D random walk, $\boldsymbol{E}x^2(t) = ct$. On a Cayley tree the walker returns to the root at time instants T_n with $\boldsymbol{E}T_n < \infty$, $n = 1, 2, \ldots$, see the explicit formulae in Corollary 2. If $\operatorname{Var} T_1 \ll (\boldsymbol{E}T_1)^2$, then the returns to the root in the logarithmic scale take place almost periodically, $\ln T_n \sim n \ln(1 + \frac{1}{2}\boldsymbol{E}T_1)$, as $n \to \infty$. On the other hand, if

$\mathrm{Var}\,T_1 \gg (\boldsymbol{E}T_1)^2$, then the periodicity in the logarithmic scale turns into chaotic behaviour without any visible pattern.

While in the latter case one can not rule out the monotonic increase of $\boldsymbol{E}x^2(t)$, in the former case one certainly has a nearly periodic vanishing of $\boldsymbol{E}x^2(e^t)$. If the magnitudes of $\mathrm{Var}\,T_1$ and $(\boldsymbol{E}T_1)^2$ are comparable one should have an intermediate situation with visible deviations in the shape of $\boldsymbol{E}x^2(t)$ from a classic ct^γ behaviour. As we approach the critical point $q + 2p = 1$, the variance

$$\mathrm{Var}\,T_1 \sim \frac{4(1-q)}{[1-(q+2p)]^3},$$

grows faster than

$$(\boldsymbol{E}T_1)^2 = \frac{4}{[1-(q+2p)]^2},$$

and we lose completely traces of the log-periodic behaviour.

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